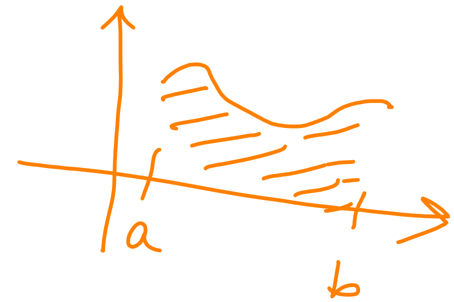
12. Demostrar que

$$\int_a^b f = \int_a^b f(x)\,dx = \int_{a+c}^{b+c} f(x-c)\,dx.$$

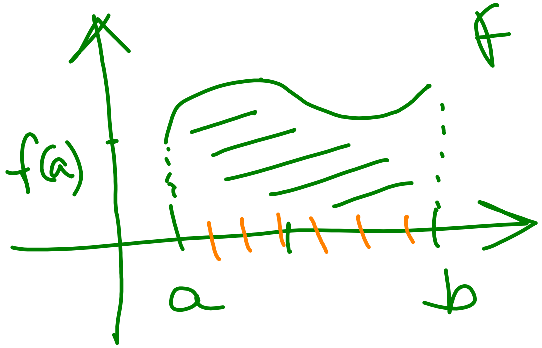


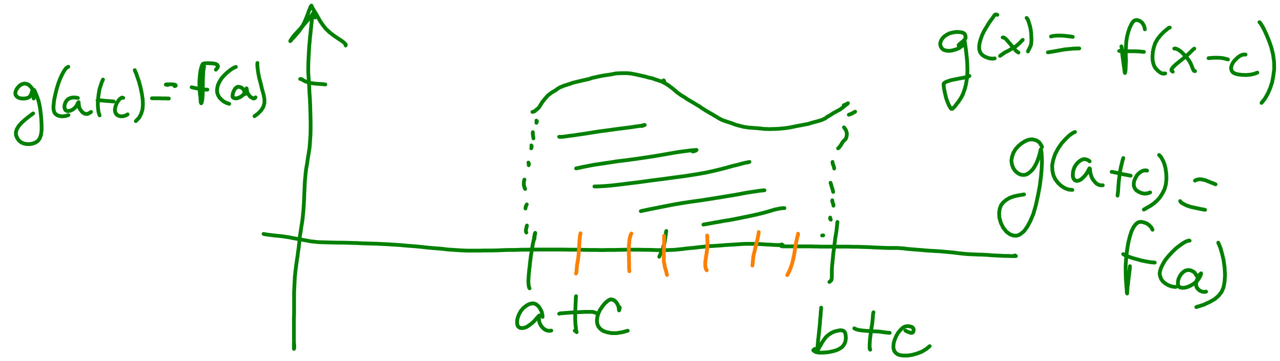
Dem: Sea $f:[a,b]\to\mathbb{R}$ integrable y sea $c\in\mathbb{R}$.

Definamos a la función $g:[a+c,b+c]\to\mathbb{R}$ como $g(x):=f(x-c)$.

Observemos que si $x\in[a+c,b+c]$, entonces $a+c\le x\le b+c$ y, por tanto, $a\le x-c\le b$.

Entonces $\underline{g(x)} = \underline{f(x-c)}$ está bien definido.





$$\int_a^b f \quad = \quad \int_{a+c}^{b+c} g \, .$$

Veamos que, en efecto, $\int_a^b f = \int_{a+c}^{b+c} g$

Probemos primero que g es integrable en su dominio usando el criterio de Riemann.

Sea $\varepsilon > 0$.

P.D: $\exists\, \tilde{P} \in \mathcal{P}[a+c, b+c]$ t.q.

$$S(\tilde{P}, g) - I(\tilde{P}, g) < \varepsilon .$$

Como f es int. en $[a,b]$, entonces existe $P \in \mathcal{P}[a,b]$, t.q. $P = \{x_0, x_1, .., x_n\}$ y

$$\boxed{S(P,f) - I(P,f) < \varepsilon .}$$

Definimos $\tilde{P} = \{x_0+c, x_1+c, x_2+c, \dots, x_n+c\}$.

Notemos que $x_0+c = a+c$, $x_n+c = b+c$ y

$$x_0+c < x_1+c < x_2+c < \dots < x_n+c$$

$\therefore \tilde{P} \in \mathcal{P}[a+c, b+c]$.

Observemos también que si

$$M_i = \sup\{f(x) : x \in [x_{i-1}, x_i]\} \text{ y si}$$

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}, \text{ entonces.}$$

$$S(P,f) - I(P,f) = \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) < \varepsilon.$$

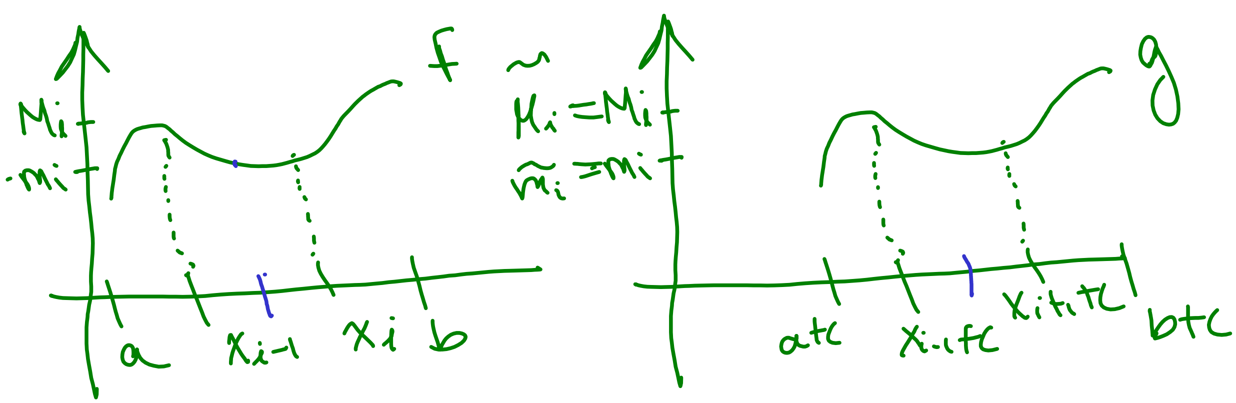
Por otro lado, si llamamos, $\tilde{x}_i := x_i + c$. y

$$\tilde{M}_i := \sup\{g(x) : x \in [\tilde{x}_{i-1}, \tilde{x}_i]\},$$

$$\tilde{m}_i := \inf\{g(x) : x \in [\tilde{x}_{i-1}, \tilde{x}_i]\}.$$

Observemos que:

$$\tilde{M}_i = \sup\{f(x-c) : x \in [x_{i-1}+c, x_i+c]\}$$



$$x \in [x_{i-1}+c, x_i+c]$$

$$\iff x_{i-1}+c \leq x \leq x_i + c$$

$$\iff x_{i-1} \leq x-c \leq x_i$$

$$\iff x-c \in [x_{i-1}, x_i].$$

$$\therefore \tilde{M}_i = \sup\{f(x-c) : x-c \in [x_{i-1}, x_i]\}.$$

$$= \sup\{f(y) : y \in [x_{i-1}, x_i]\}.$$

$$= M_i$$

$\star\star$

De manera análoga, $\tilde{m}_i = m_i$

Escribimos ahora

$$S(\tilde{P}, g) - I(\tilde{P}, g) = \sum_{i=1}^{n} (\tilde{M}_i - \tilde{m}_i)(\tilde{x}_i - \tilde{x}_{i-1})$$

$$= \sum_{i=1}^{n} (M_i - m_i)\left(x_i + c - (x_{i-1} + c)\right)$$

$$= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}).$$

$$= S(P,f) - I(P,f) < \varepsilon . \; \text{-----} \; \star$$

$\therefore$ g es integrable y si P satisface $(*)$, ent,

$$S(\tilde{P}, g) - I(\tilde{P}, g) < \varepsilon \quad \text{con } \tilde{P} = \{x_0 + c, \dots, x_n + c\}.$$

Entonces, dada $\varepsilon > 0$, y tomando P y $\tilde{P}$ como arriba,

$$\begin{aligned}
\int_a^b f &\leq S(P, f) \\
&= S(\tilde{P}, g) \qquad \text{por } (**) \\
&< I(\tilde{P}, g) + \varepsilon. \\
&\leq \int_{a+c}^{b+c} g + \varepsilon.
\end{aligned}$$

Tomando lím cuando $\varepsilon \to 0$, concluimos que $\int_a^b f \leq \int_{a+c}^{b+c} g$.

De manera totalmente análoga se prueba que

$$\int_{a+c}^{b+c} g \leq \int_a^b f + \varepsilon \quad \forall \varepsilon > 0.$$

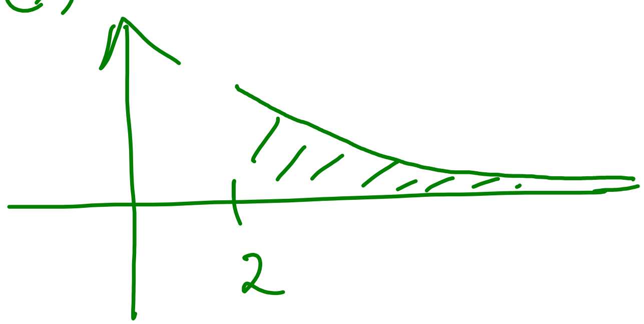
$$\therefore \int_a^b f = \int_{a+c}^{b+c} g = \int_{a+c}^{b+c} f(x-c)\, dx.$$

$\blacksquare$

(vii) bis $\displaystyle\int_2^\infty \frac{1}{x \cdot \log^2(x)}\, dx.$

Nótese que $\displaystyle\lim_{x\to\infty} \frac{1}{x \log^2(x)} = 0$ y

$\dfrac{1}{x \log^2(x)}$ es acotada en $[2, \infty)$.



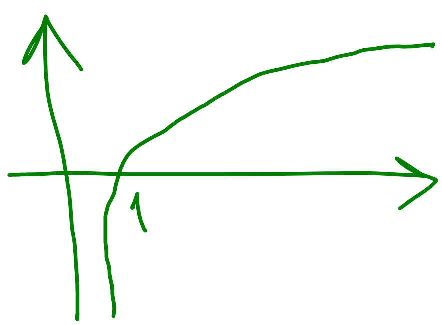
Buscamos ver si $\exists \lim_{c \to \infty} \int_2^c \frac{1}{x \log^2(x)} dx$.

Dado $c > 2$,

$$\int_2^c \frac{1}{x \cdot \log^2(x)} dx \underset{\substack{y = \log(x) \\ dy = \frac{1}{x} dx}}{=} \int_{\log(2)}^{\log(c)} \frac{dy}{y^2} = -\frac{1}{y}\Big|_{\log(2)}^{\log(c)}$$

$$= -\frac{1}{\log(c)} + \frac{1}{\log(2)}.$$

Luego, $\lim_{c \to \infty} \int_2^c \frac{1}{x \log^2(x)} dx = \lim_{c \to \infty} \left(-\frac{1}{\log(c)} + \frac{1}{\log(2)} \right).$



$$= 0 + \frac{1}{\log(2)} = \frac{1}{\log(2)}$$

$$\therefore \int_2^\infty \frac{1}{x \log^2(x)} dx = \frac{1}{\log(2)}.$$

La justificación para la integral

$$\int_2^c \frac{1}{x\cdot\log^2(x)}\,dx$$ usando el T. cambio de variable, sería escribiendo

$$\frac{1}{x\cdot\log^2(x)} = g\circ\varphi(x)\cdot\varphi'(x)$$

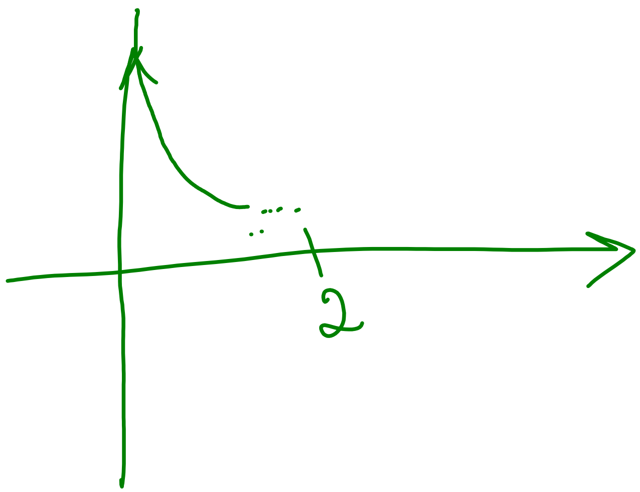
con $\varphi(x) = \log(x)$ y $g(y) = \frac{1}{y^2}$.

$$\therefore\ g(\varphi(x))\cdot\varphi'(x) = \frac{1}{\log^2(x)}\cdot\frac{1}{x}.$$

$$\therefore\ \int_2^c g(\varphi(x))\cdot\varphi'(x)\,dx = \int_{\varphi(2)}^{\varphi(c)} g(y)\,dy$$

20 (iii) $\displaystyle\int_1^2 \frac{x}{\sqrt{x-1}}\,dx$.

Nótese que $\displaystyle\lim_{x\to 1^+} \frac{x}{\sqrt{x-1}} \to +\infty$



Si $c \in (1,2)$, ent.

$$\int_c^2 \frac{x}{\sqrt{x-1}}\,dx \underset{\substack{y=x-1\\ y+1=x\\ dy=dx}}{=}$$

$$= \int_{c-1}^1 \frac{y+1}{\sqrt{y}}\,dy = \int_{c-1}^1 \left(\frac{y}{\sqrt{y}} + \frac{1}{\sqrt{y}}\right)dy =$$

$$\overset{y>0}{=} \int_{c-1}^1 \left(\sqrt{y} + \frac{1}{\sqrt{y}}\right)dy = \left(\frac{2}{3}y^{3/2} + 2y^{1/2}\right)\Big|_{c-1}^{1} =$$

$$\left(\frac{2}{3}+2\right)-\left(\frac{2}{3}(c-1)^{3/2}+2(c-1)^{\frac{1}{2}}\right)=$$

$$\frac{8}{3}-\frac{2}{3}(c-1)^{3/2}-2(c-1)^{\frac{1}{2}}.$$

$$\lim_{c\to 1}\int_c^2\frac{x}{\sqrt{x-1}}\,dx=\lim_{c\to 1}\frac{8}{3}-\frac{2}{3}(c-1)^{3/2}-2(c-1)^{\frac{1}{2}}$$

$$=\frac{8}{3}=\int_1^2\frac{x}{\sqrt{x-1}}\,dx.$$

Con T. de C de V:

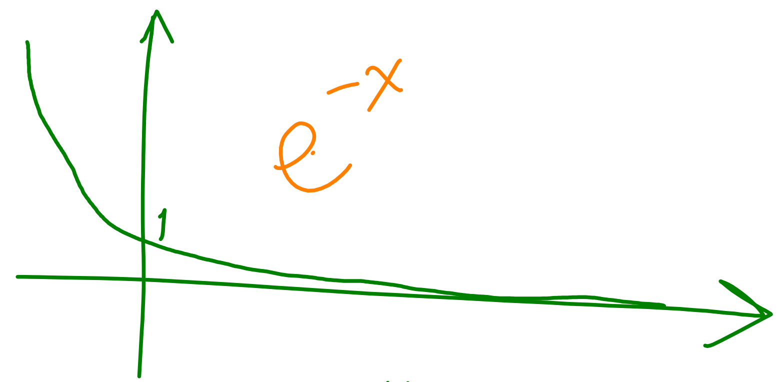
$$\frac{x}{\sqrt{x-1}}=g(\varphi(x))\cdot\varphi'(x)$$

con $\varphi(x)=x-1$, $\varphi'(x)=1$

$$g(y)=\frac{y+1}{\sqrt{y}}$$

$$\int_c^2 g(\varphi(x))\cdot\varphi'(x)\,dx \underset{T.C.V.}{=} \int_{\varphi(c)}^{\varphi(2)} g(y)\,dy.$$

(v) $\displaystyle\int_0^\infty e^{-x}\,dx.$



Dado $K>0$:

$$\int_0^K e^{-x}\,dx = -e^{-x}\Big|_0^K = -e^{-K} + e^0$$

$$= 1 - e^{-K}.$$

$$\therefore \lim_{K\to\infty}\int_0^K e^{-x}\,dx = \lim_{K\to\infty} 1 - e^{-K} = 1$$

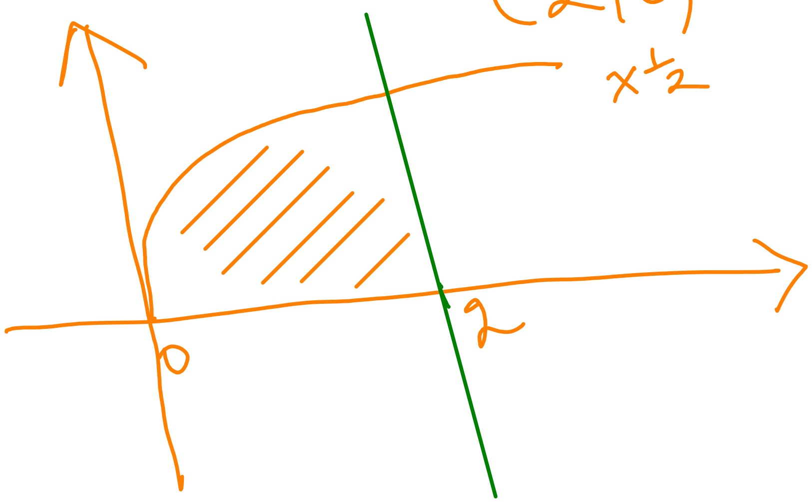
$$\therefore \int_0^\infty e^{-x}\,dx = 1.$$

11 (ν) Área entre las curvas:

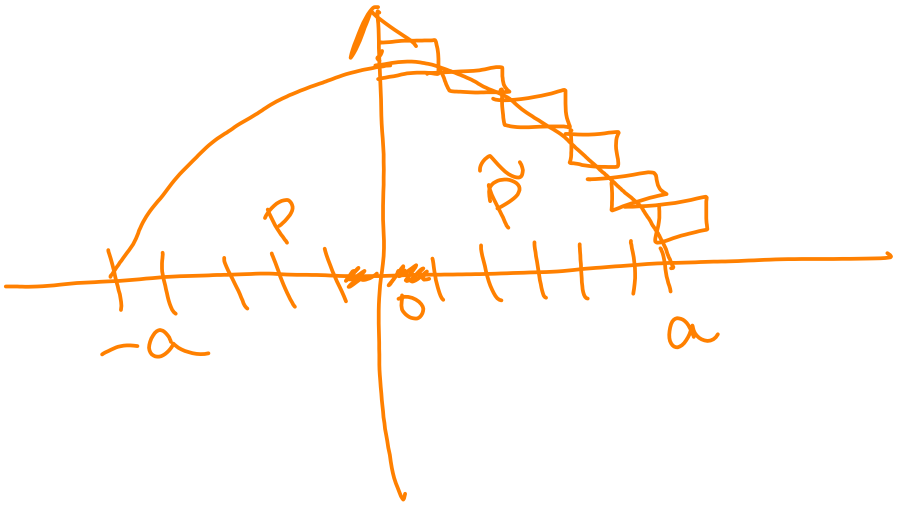
$f(x) = x^{\frac{1}{2}}$

Eje $\overline{\underline{X}}$

La recta vertical que pasa por $(2,0)$



8. PD: $\displaystyle\int_{-a}^{0} f = \int_{0}^{a} f$.



Sugerencia: tomen P partición de $[-a,0]$ tal que para una partición $\tilde{P}$ correspondiente a $[0,a]$

$$\longrightarrow S(f_{[0,a]},\tilde{P}) - \bar{I}(f_{[0,a]},\tilde{P}) < \varepsilon.$$

Comparen $S(f_{[0,a]},\tilde{P})$ con $S(f_{[-a,0]},P)$.