Tarea 5.

2(i). Sea $P(x) = x^2 - 4x - 9$.

Sea $x_0 = 3$.

Calcular $P_{2,3}(x)$:

Sabemos que $P'(x) = 2x - 4$.

$P''(x) = 2$.

$$P(3) = 3^2 - 4 \cdot 3 - 9 = -12$$

$$P'(3) = 2 \cdot 3 - 4 = 2$$

$$P''(3) = 2.$$

Luego,

$$P_{2,3}(x) = P(3) + P'(3)(x-3) + \frac{P''(3)}{2}(x-3)^2.$$

$$= -12 + 2(x-3) + (x-3)^2.$$

Por Teo. de clase, $P(x)$, visto como función, es igual a $P_{2,3}(x)$ hasta el orden 2. Por tanto,

$P(x) = P_{2,3}(x)$ pues dos polinomios de grado n que son iguales hasta el orden n, deben ser el mismo. //

Para el 5 (i):
Calcular sen(1) con error $< 10^{-17}$.

---

6 (i) P.D. $$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}.$$

Sabemos que:

$$\operatorname{sen}(x+y) = \operatorname{sen}(x)\cdot\cos(y) + \operatorname{sen}(y)\cdot\cos(x).$$

$$\cos(x+y) = \cos(x)\cos(y) - \operatorname{sen}(x)\cdot\operatorname{sen}(y).$$

$$\tan(x+y) = \frac{\operatorname{sen}(x+y)}{\cos(x+y)} = \frac{\operatorname{sen}(x)\cdot\cos(y) + \operatorname{sen}(y)\cos(x)}{\cos(x)\cos(y) - \operatorname{sen}(x)\cdot\operatorname{sen}(y)}.$$

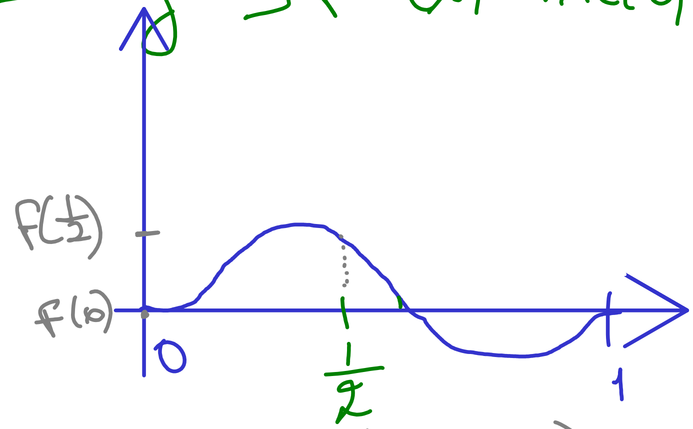
$$= \frac{\dfrac{\operatorname{sen}(x)\cos(y)}{\cos(x)\cos(y)} + \dfrac{\operatorname{sen}(y)\cos(x)}{\cos(x)\cos(y)}}{1 - \dfrac{\operatorname{sen}(x)\operatorname{sen}(y)}{\cos(x)\cos(y)}} = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}$$

9. Sup. que $f:[0,1]\to\mathbb{R}$ es t.q.

(i) f y f' son continuas en $[0,1]$ y $\exists f''(x)\ \forall x\in(0,1)$;

(ii) $f'(0)=f'(1)=0$

(iii) $|f''(x)|\leq 1 \quad \forall x\in(0,1)$.



P.D. $\left|f\left(\tfrac{1}{2}\right)-f(0)\right|\leq \frac{1}{8}$. $\left(\text{y } |f(1)-f(0)|\leq \frac{1}{4}\right)$

Dems. Por Teo. de Taylor, sabemos que

$$f(x)=f(0)+f'(0)x+\frac{f''(\xi_x)}{2}x^2 \qquad \forall x\in[0,1],$$

con $\xi_x\in(0,x)$.

entonces:

$$f\left(\tfrac{1}{2}\right)=f(0)+\cancelto{0}{f'(0)}\cdot\frac{1}{2}+\frac{f''(\xi_{\frac{1}{2}})}{2}\cdot\frac{1}{4}\ ;\quad \xi_{\frac{1}{2}}\in\left(0,\tfrac{1}{2}\right).$$

$$\therefore \left|f\left(\tfrac{1}{2}\right) - f(0)\right| = \left|\frac{f''(\xi_{\frac{1}{2}})}{2}\right| \cdot \frac{1}{4} \underset{(iii)}{\leq} \frac{1}{8}.$$

pues $\xi_{\frac{1}{2}} \in \left(0, \tfrac{1}{2}\right)$. //

13 (ii) Sea $h(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{si } x \neq 0 \\ 0 & \text{si } x = 0. \end{cases}$

P.D. $h^{(n)}(0) = 0 \quad \forall n \in \mathbb{N}$.

Para $n=0$, $h(0) = 0$ por def.

Luego, para $x \neq 0$:

$$h'(x) = e^{-\frac{1}{x^2}} \left(-\frac{x^{-3}}{-2}\right) = e^{-\frac{1}{x^2}} \cdot \frac{1}{2x^3}.$$

$$y \quad h'(0) = \lim_{x\to 0} \frac{h(x) - h(0)}{x} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} - 0}{x} \underset{(*)}{=} 0$$

(*) Por Lema: que dice que $\lim_{x\to 0} \frac{h(x)}{x^k} = 0 \quad \forall k \in \mathbb{N}$.

Para calcular $h''(0)$:

Tomamos

$$h''(0) = \lim_{x\to 0} \frac{h'(x) - h'(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} \cdot \frac{1}{2x^3}}{x} =$$

$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{2x^4} \underset{\text{Lema}}{=} 0.$$

Para calcular $h^{(n)}(0)$, procedemos inductivamente:

(I) Para $n=0$, $n=1$, $n=2$, ya lo tenemos.

(II) Sup que $h^{(k)}(0) = 0$.

PD $h^{(k+1)}(0) = 0$.

Sabemos que $h'(x) = \begin{cases} e^{-\frac{1}{x^2}} \cdot \frac{1}{2x^3} & \text{si } x \neq 0. \\ 0 & \text{si } x = 0. \end{cases}$

Afirmamos que $\forall n \geq 1$, $\boxed{h^{(n)}(x) = e^{-\frac{1}{x^2}} \cdot Q(x)}$ (*)

con $Q(x) = \sum_{j=1}^{m} a_j \cdot \frac{1}{x^j}$. con $a_j \in \mathbb{R}$.

Dando esto por hecho por un momento, se tiene entonces

$$h^{(k+1)}(0) = \lim_{x\to 0} \frac{h^{(k)}(x) - h^{(k)}(0)}{x - 0}$$

$$\underset{\substack{H.I \\ h^{(k)}(0)=0.}}{=} \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}} \cdot Q(x) - 0}{x}$$

$$= \lim_{x\to 0} e^{-\frac{1}{x^2}} \left(\sum_{j=1}^{m_k} a_j \cdot \frac{1}{x^{j+1}} \right)$$

$$= \lim_{x\to 0} \left(\sum_{j=1}^{m_k} a_j \cdot e^{-\frac{1}{x^2}} \cdot \frac{1}{x^{j+1}} \right)$$

$$\underset{\text{Lema}}{=} 0.$$

Resta demostrar:

(I) $$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^k} = 0 \quad \forall k \geq 1.$$

(II) $$h^{(n)}(x) = e^{-\frac{1}{x^2}} \cdot \left(\sum_{j=1}^{m_n} a_j \cdot \frac{1}{x^j} \right) \cdot \quad \forall n \geq 1. \quad \forall x \neq 0$$

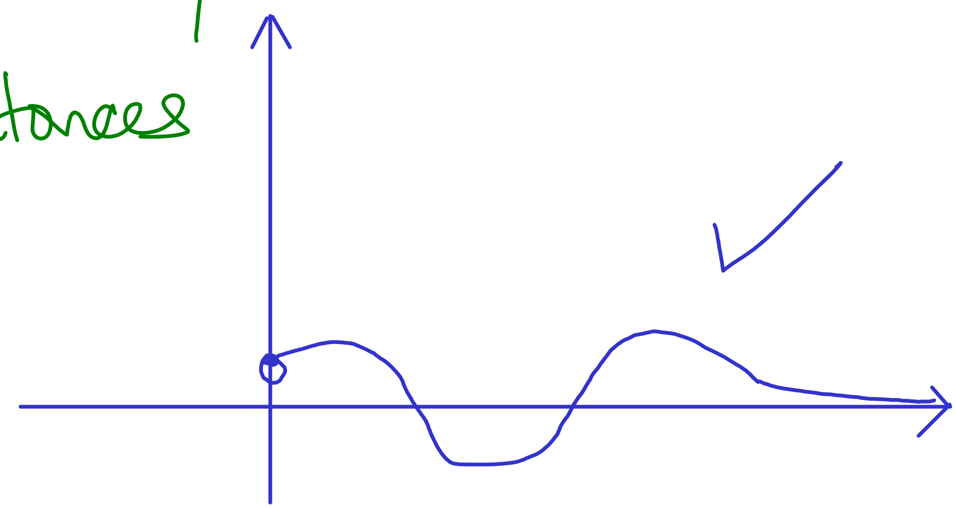
(Mostrar esto por inducción).

11. (iii) Demostrar que si f es dos veces derivable en $(0,\infty)$, f'' es acotada y

$$\lim_{x\to\infty} f(x) = 0,$$

entonces

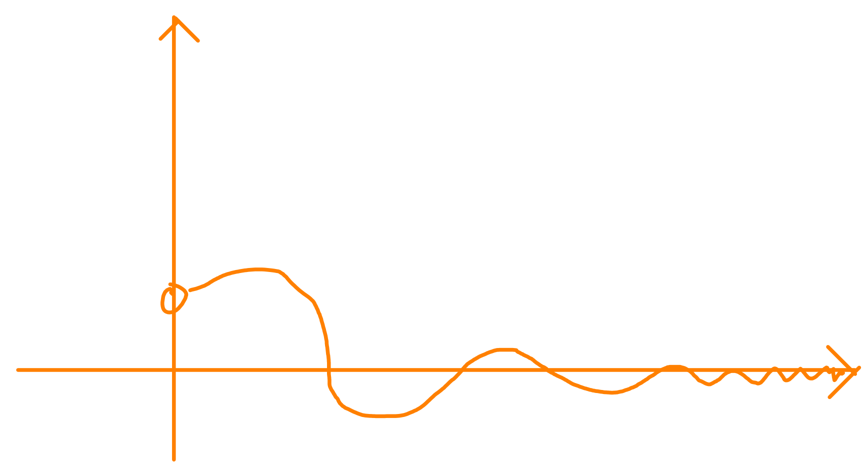
$$\lim_{x\to\infty} f'(x) = 0.$$



Recordatorio:

Si $g:(a,\infty) \longrightarrow \mathbb{R}$,

decimos que $\lim_{x\to\infty} g(x) = 0$ sii



$$\forall \varepsilon > 0 \ \exists K > 0 \ (x > K \Rightarrow |g(x)| < \varepsilon).$$

11 (i). Sup que $f:(0,\infty)\longrightarrow\mathbb{R}$ es dos veces derivable y que M_0 y M_2 son constantes positivas tales que:

(a) $|f(x)|\leq M_0 \quad \forall x>0$.

(b) $|f''(x)|\leq M_2 \quad \forall x>0$.

Demostrar que $\forall x>0$ y $\forall h>0$ se cumple que

$$|f'(x)|\leq \frac{2}{h}M_0+\frac{h}{2}M_2.$$

P.D. Sean $x>0$ y $h>0$ arbitrarias: Ent.

$$f(x+h)=f(x)+f'(x)(x+h-x)+\frac{f''(\xi_{x+h})}{2}(x+h-x)^2$$

(por Teo. de Taylor para $n=1$).

Esto implica que

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{f''(\xi_{x+h})}{2}\cdot h^2.$$

Entonces:

$$|f'(x)| = \left| \frac{f(x+h) - f(x)}{h} - \frac{f''(\xi_{x+h})}{2h}\cdot h^2 \right|.$$

$$\underset{\substack{h>0 \text{ y} \\ \text{des. del } \Delta.}}{\leq} \frac{|f(x+h)| + |f(x)|}{h} + \frac{|f''(\xi_{x+h})|}{2}\, h.$$

$$\leq \boxed{\frac{2M_0}{h} + \frac{M_2}{2}h}.$$

l.q.q.d.

(ii) Bajo las hip. de (i), mostrar que

$$|f'(x)| \leq 2\sqrt{M_0 \cdot M_2}. \quad \forall x > 0.$$

Sabemos que: $\forall h > 0 \quad \forall x > 0.$

$$|f'(x)| \leq \frac{2M_0}{h} + \frac{M_2 h}{2} \underset{???}{\leq} 2\sqrt{M_0 \cdot M_2}.$$

Obtener h tal que se cumple.

"Haciendo las cuentas "en reversa"":

$$\left(\frac{2M_0}{h}+\frac{M_2 h}{2}\right) \le 2\sqrt{M_0 M_2} \underset{\substack{\text{pues}\\ \text{todo es}\\ \geq 0.}}{\iff} \left(\frac{2M_0}{h}+\frac{M_2 h}{2}\right)^2 \le 4 M_0 M_2$$

$$\iff \frac{2^2 M_0^2}{h^2} + \cancel{2}\cdot\frac{2M_0}{\cancel{h}}\cdot\frac{M_2 \cancel{h}}{\cancel{2}} + \frac{M_2^2 h^2}{2^2} \le 4 M_0 M_2 .$$

$$\iff \frac{4M_0^2}{h^2} + 2M_0M_2 - 4M_0M_2 + \frac{M_2^2 h^2}{4} \le 0 .$$

$$\iff \frac{4M_0^2}{h^2} - 2M_0M_2 + \frac{M_2^2 h^2}{4} \le 0 .$$

$$\iff \left(\frac{2M_0}{h} - \frac{M_2 h}{2}\right)^2 \le 0 .$$

$$\Longleftrightarrow \left(\frac{2M_0}{h} - \frac{M_2 h}{2}\right)^2 = 0.$$

pues $y^2 \geq 0$ $\forall y \in \mathbb{R}$.

$$\Longleftrightarrow \frac{2M_0}{h} - \frac{M_2 h}{2} = 0.$$

$$\Longleftrightarrow \frac{2M_0}{h} = \frac{M_2 h}{2} \Longleftrightarrow \frac{4M_0}{M_2} = h^2.$$

$$\Longleftrightarrow \frac{2\sqrt{M_0}}{\sqrt{M_2}} = h$$

Dems de (ii): Sea $h = \dfrac{2\sqrt{M_0}}{\sqrt{M_2}}$

Ent .....

$$\therefore \frac{2M_0}{h} + \frac{M_2 h}{2} = 2\sqrt{M_0}\cdot\sqrt{M_2}.$$

$$h = 2\,\frac{\sqrt{M_0 M_2}}{M_2}$$

$$= \frac{2\sqrt{M_0}\cdot\cancel{\sqrt{M_2}}}{\sqrt{M_2}\,\cancel{\sqrt{M_2}}}$$

(iii) Sup. que $f:(0,\infty) \to \mathbb{R}$ es dos veces derivable y que $|f''(x)| \leq M_2 \quad \forall x > 0$.

Sup. además que $\lim\limits_{x\to\infty} f(x) = 0$.

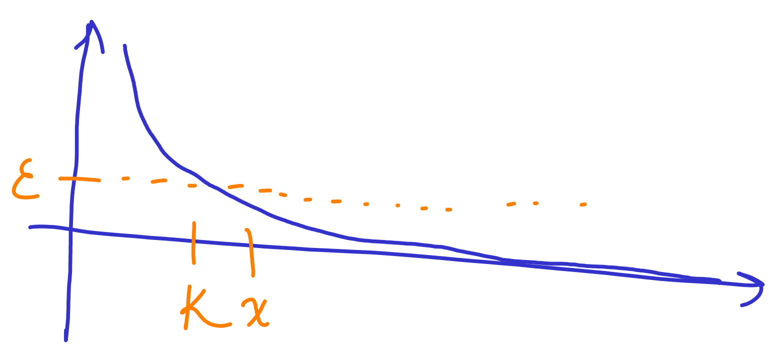
P.D. $\lim\limits_{x\to\infty} f'(x) = 0$.

Por hip. tenemos que:

$$\forall \varepsilon > 0 \ \exists K > 0 \ (x > K \Rightarrow |f(x)| < \varepsilon).$$

P.D.:

$$\forall \tilde{\varepsilon} > 0 \ \exists \tilde{K} > 0 \ (x > \tilde{K} \Rightarrow |f'(x)| < \tilde{\varepsilon}).$$



Sea $\tilde{\varepsilon}>0$ arbitraria

Sea $\tilde{K}>0$ t.q. si $\varepsilon := \dfrac{\tilde{\varepsilon}^2}{4M_2}$, entonces, $\tilde{K}=K$

con:

$$K>0 \text{ t.q. } \left(x>K \Rightarrow |f(x)|<\varepsilon\right).$$

Sea $x>K=\tilde{K}$.

Por Teo. de Taylor, $\forall h>0$ se cumple:

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{f''(\xi_{x+h})}{2}\cdot h^2$$

con $\xi_{x+h} \in (x, x+h)$.

y por tanto:

$$|f'(x)| = \left|\frac{f(x+h-f(x)}{h} - \frac{f''(\xi_{x+h})}{2}h\right|$$

$$\leq \frac{|f(x+h)| + |f(x)|}{h} + \frac{|f''(\xi_{x+h})|}{2} h \quad \ldots\ldots \; (*)$$

Nótese que como $h > 0$, ent $x+h > x > K$.

$$\therefore \; (*) < \frac{\varepsilon + \varepsilon}{h} + \frac{M_2}{2} h = \frac{2\varepsilon}{h} + \frac{M_2 h}{2}.$$

Al igual que en (ii), si $h = \dfrac{2\sqrt{\varepsilon}}{\sqrt{M_2}}$,

esto nos lleva a que

$$\frac{2\varepsilon}{h} + \frac{M_2 h}{2} = 2\sqrt{\varepsilon} \cdot \sqrt{M_2}.$$

Entonces: $\forall x > K$,

$$|f'(x)| < 2\sqrt{\varepsilon} \cdot \sqrt{M_2} = \tilde{\varepsilon} \quad \text{si}$$

$\sqrt{\varepsilon} = \dfrac{\tilde{\varepsilon}}{2\sqrt{M_2}}$, es decir, si $\boxed{\varepsilon = \dfrac{\tilde{\varepsilon}^2}{4M_2}.}$

$\therefore \forall x \left(x > \tilde{K} \Rightarrow |f'(x)| < \tilde{\varepsilon}\right)$

$\therefore \lim_{x\to\infty} f'(x) = 0.$

$\blacksquare$

1 (iv). Calcular pol. de Taylor de grado $2n$ en $\frac{\pi}{2}$

para $f(x) = \operatorname{sen}(x)$.

(Recordemos que $P_{2n+1,0}(x) = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots + \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$.

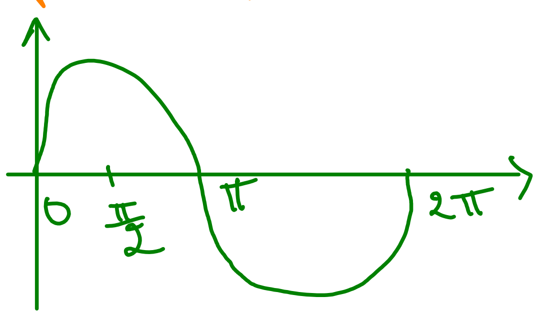
$f(x) = \text{sen}(x)$

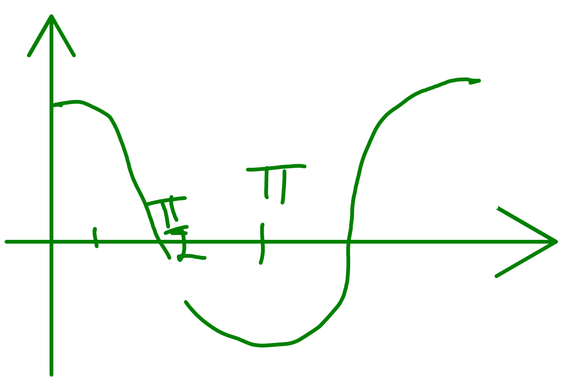
$f'(x) = \cos(x)$

$f''(x) = -\text{sen}(x)$

$f'''(x) = -\cos(x)$

$f^{(iv)}(x) = \text{sen}(x)$





$f\left(\frac{\pi}{2}\right) = 1$

$f'\left(\frac{\pi}{2}\right) = 0$

$f''\left(\frac{\pi}{2}\right) = -1$

$f'''\left(\frac{\pi}{2}\right) = 0$

$f^{(iv)}\left(\frac{\pi}{2}\right) = 1$

$$P_{4,\frac{\pi}{2}}(x) = f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \dots + \frac{f^{(iv)}\left(\frac{\pi}{2}\right)}{4!}\left(x - \frac{\pi}{2}\right)^4.$$