Tarea 6

1 (ii) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2-1}}$.

Obs: $\forall n \geq 2$ $\dfrac{1}{\sqrt[3]{n^2-1}}$ está bien definido.

$$\frac{1}{\sqrt[3]{n^2-1}} \approx \frac{1}{n^{2/3}}$$ cuando n es "grande"

Sabemos que $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{2/3}}$ es divergente.

$$\frac{1}{n^{2/3}} \leq \frac{1}{(n^2-1)^{\frac{1}{3}}}.$$ $\forall n \geq 2$ pues

$\forall n \geq 2.$ $n^2-1 \leq n^2$ y por tanto, $(n^2-1)^{\frac{1}{3}} \leq n^{2/3}$.

$\therefore$ Por el corolario al crit. de comp. (I), se sigue que

$$\sum_{n=2}^{\infty} \frac{1}{(n^2-1)^{\frac{1}{3}}}$$ es divergente

(i) $$\sum_{n=1}^{\infty} \frac{\operatorname{sen}(n\theta)}{n^2}$$ ¿Es convergente?

<u>Af:</u> $$\sum_{n=1}^{\infty} \frac{|\operatorname{sen}(n\theta)|}{n^2}$$ converge.

Como $\forall n \geq 1$ $\quad 0 \leq \frac{|\operatorname{sen}(n\theta)|}{n^2} \leq \frac{1}{n^2}$.

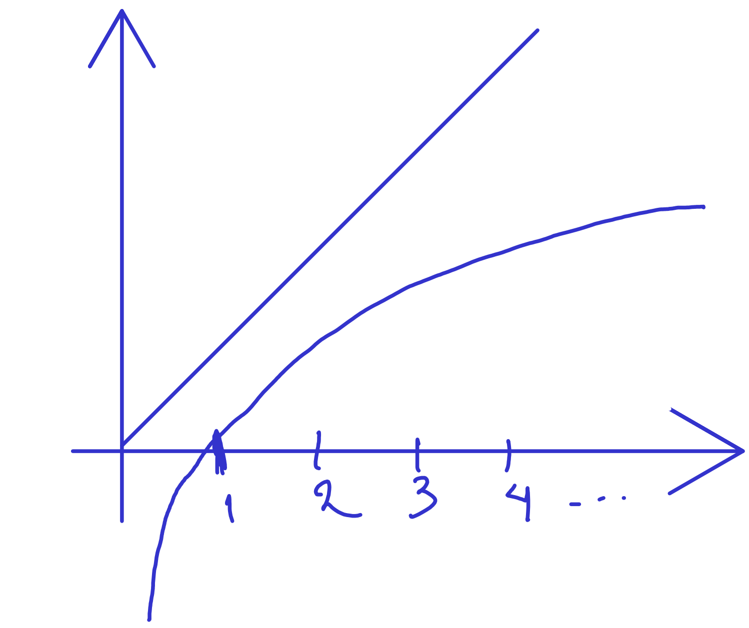
Como $\sum_{n=1}^{\infty} \frac{1}{n^2}$ convergente, ent. por C. de C. (I).

se sigue que $$\sum_{n=1}^{\infty} \frac{|\operatorname{sen}(n\theta)|}{n^2}$$ converge.

∴ Por el Teo. de convergencia absoluta, ent.

$$\sum_{n=1}^{\infty} \frac{\operatorname{sen}(n\theta)}{n^2} \quad \text{converge.}$$

(iv) $$\sum_{n=2}^{\infty} \frac{1}{\log(n)}.$$



$$\boxed{\underline{\text{Af.}} \quad \log(n) < n \quad \forall n \geq 2.}$$

Si esto es cierto, entonces

$$\frac{1}{n} < \frac{1}{\log(n)} \quad \forall n \geq 2.$$

y, por tanto, como la serie armónica diverge, ent. por C.C.I,

$$\sum_{n=2}^{\infty} \frac{1}{\log(n)} \quad \text{diverge.}$$

Veamos que para n suficientemente grande, $\dfrac{n}{\log(n)} > 1.$

Nótese que

$$\lim_{x\to\infty} \frac{\log(x)}{x} \overset{?}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0.$$

$$\Rightarrow \lim_{x\to\infty} \frac{x}{\log(x)} = \infty \rightsquigarrow \forall M>0\ \exists K\in\mathbb{N}\ \forall x \left(x>K \Rightarrow \frac{x}{\log(x)} > M \right)$$

Por tanto, $\exists K\in\mathbb{N}\ \forall n\geq K\quad \frac{n}{\log(n)} > 1.$

$\therefore \forall n\geq K \quad n > \log(n)$ y esto prueba la afirmación. //

Como $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge, entonces la sucesión

$$S_N = \sum_{n=1}^{N} \frac{1}{n} \text{ no es acotada superiormente.}$$

Ent. $\forall M > 0 \quad \exists N \in \mathbb{N}$ t.q.

$$M < S_N = \sum_{n=1}^{N} \frac{1}{n}$$

$$= \sum_{n=1}^{K} \frac{1}{n} + \sum_{n=K+1}^{N} \frac{1}{n}$$

$$< \sum_{n=1}^{K} \frac{1}{n} + \sum_{n=K+1}^{N} \frac{1}{\log(n)}.$$

$$= \sum_{n=1}^{K} \left(\frac{1}{n} - \frac{1}{\log(n)} \right)$$

$$+ \sum_{n=1}^{K} \frac{1}{\log(n)} + \sum_{n=K+1}^{N} \frac{1}{\log(n)}$$

$$= \text{Constante}(K) + \sum_{n=1}^{N} \frac{1}{\log(n)}.$$

$$\Rightarrow \quad M - Cte(K) < \sum_{n=1}^{N} \frac{1}{\log(n)}.$$

$$\therefore \ \forall M > 0 \qquad M - Cte(K) < \sum_{n=1}^{N} \frac{1}{\log(n)}$$

$\tilde{S}_N = \sum_{n=1}^{N} \frac{1}{\log(n)}$ no es acotada superiormente

1.(vii) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$

Notemos que $\displaystyle\lim_{n\to\infty} \frac{\frac{n^2}{n^3+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+1} = 1.$

$\therefore$ Por C.C.II (Criterio del límite), como $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

diverge, ent. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$ diverge. $\lrcorner$

Ent, si $S_N = \displaystyle\sum_{n=1}^{N} \frac{n^2}{n^3+1}$, $(S_N)_{N=1}^{\infty}$ diverge,

entonces $\tilde{S}_N := \displaystyle\sum_{n=0}^{N} \frac{n^2}{n^3+1} = 0 + \sum_{n=1}^{N} \frac{n^2}{n^3+1}$ diverge.

$\therefore \displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+1}$ diverge.

4. (a) Sea $p \in \mathbb{N}^+$. Demostrar que

$$\sum_{k=1}^{\infty} \frac{1}{k(k+p)} \quad \text{converge a} \quad \left(\frac{1}{p}\sum_{j=1}^{p} \frac{1}{j}\right).$$

Dems. Si $n \in \mathbb{N}^+$

$$S_n = \sum_{k=1}^{n} \frac{1}{k(k+p)} = \frac{1}{p}\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+p}\right) =$$

$$\frac{1}{p}\left[\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+p}\right] \underset{\text{si } n>p}{=}$$

$$\frac{1}{p}\left[\sum_{k=1}^{p} \frac{1}{k} + \sum_{k=p+1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+p}\right] =$$

$$\frac{1}{p}\left[\sum_{k=1}^{p}\frac{1}{k}+\sum_{k=p+1}^{n}\frac{1}{k}-\sum_{k=p+1}^{n+p}\frac{1}{k}\right] \underset{p+1\le n<n+p}{=}$$

$$\frac{1}{p}\left[\sum_{k=1}^{p}\frac{1}{k}-\sum_{k=n+1}^{n+p}\frac{1}{k}\right]=$$

¿Qué ocurre con $\displaystyle\sum_{k=n+1}^{n+p}\frac{1}{k}=\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+p}$?

$$\forall n\geq 1: \quad 0\leq\sum_{k=n+1}^{n+p}\frac{1}{k}\leq p\cdot\frac{1}{n+1}\xrightarrow[n\to\infty]{}0.$$

Ent. $\displaystyle\lim_{n\to\infty}\sum_{k=n+1}^{n+p}\frac{1}{k}=0$

Ent. $\displaystyle\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[\frac{1}{p}\cdot\sum_{k=1}^{p}\frac{1}{k}-\sum_{k=n+1}^{n+p}\frac{1}{k}\right]=\frac{\sum_{k=1}^{p}\frac{1}{k}}{p}$

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geq N \quad \left| S_n - \frac{\sum_{k=1}^{p} \frac{1}{k}}{p} \right| < \varepsilon .$$

Sea $\varepsilon > 0$.

Necesitamos $N > p$ y N t.q. $\forall n \geq N$

$$\left| S_n - \frac{\sum_{k=1}^{p} \frac{1}{k}}{p} \right| < \varepsilon$$

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