2 (ii) Usa el crit. del cociente para ver si converge

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}.$$

Sea $a_n := \frac{n!}{n^n}$, entonces

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{n^n (n+1)!}{(n+1)^{n+1} \cdot n!} = \frac{n^n \cancel{(n+1)}}{(n+1)^n \cancel{(n+1)}}$$

$$= \left(\frac{n}{n+1}\right)^n.$$

Veremos que $\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \xrightarrow[n\to\infty]{} e.$

$$\left(1+\frac{1}{n}\right)^n = e^{\log\left(\left(1+\frac{1}{n}\right)^n\right)} = e^{n\cdot\log\left(1+\frac{1}{n}\right)}.$$

Con L'Hôpital se puede ver que $n\cdot\log\left(1+\frac{1}{n}\right)\to 1$

Sea $f(x)=x\cdot\log\left(1+\frac{1}{x}\right) = \dfrac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}}.$

$$\left\{\begin{array}{l} \text{Si } \lim\limits_{x\to\infty} x\cdot\log\left(1+\frac{1}{x}\right)=1, \text{ ent.} \\ \lim\limits_{n\to\infty} n\cdot\log\left(1+\frac{1}{n}\right)=1. \end{array}\right.$$

Notemos que: $\dfrac{\left(\frac{1}{1+\frac{1}{x}}\right)\left(-\frac{1}{x^2}\right)}{-\frac{1}{x^2}} = \dfrac{1}{1+\frac{1}{x}} \cdot \forall x\neq 0.$

Luego, como $\lim\limits_{x\to\infty}\dfrac{1}{1+\frac{1}{x}} = 1.$

Ent, por Teo. de L'Hôpital,

$$\lim_{x\to\infty} x\cdot\log\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\log\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{1+\frac{1}{x}}$$

$$= 1.$$

Como e^x es una función continua

$$\lim_{n\to\infty} e^{n\log\left(1+\frac{1}{n}\right)} = e.$$

$$\therefore \left(\frac{n}{n+1}\right)^n \xrightarrow[n\to\infty]{} \frac{1}{e} < 1$$

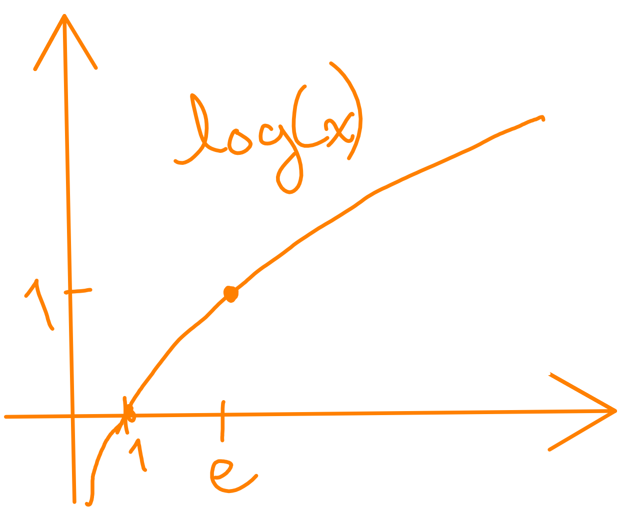
Ent, la serie $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converge.

3 (i) Sea $\alpha_n = \frac{\log(n)}{n}$ para $n \geq 1$.

Verifica si $\sum_{n=1}^{\infty} \alpha_n$ es convergente.

Sea $f(x) = \frac{\log(x)}{x}$ para $x \in [1, \infty)$.

Entonces: $f'(x) = \frac{\frac{1}{x} \cdot x - 1 \cdot \log(x)}{x^2} = \frac{1 - \log(x)}{x^2}$.



$< 0 \quad \forall x \in (e, \infty)$.

$\therefore f$ es decreciente en (e, ∞).

Obsérvese también que f es continua en (e,∞) y, por tanto

$\forall k \geq 3$ f es integrable en $[k, k+1]$.

Por tanto, por el criterio de la integral para series $\sum_{k=3}^{\infty} \frac{\log(k)}{k}$ converge sii $\int_3^{\infty} \frac{\log(x)}{x}\,dx$ converge

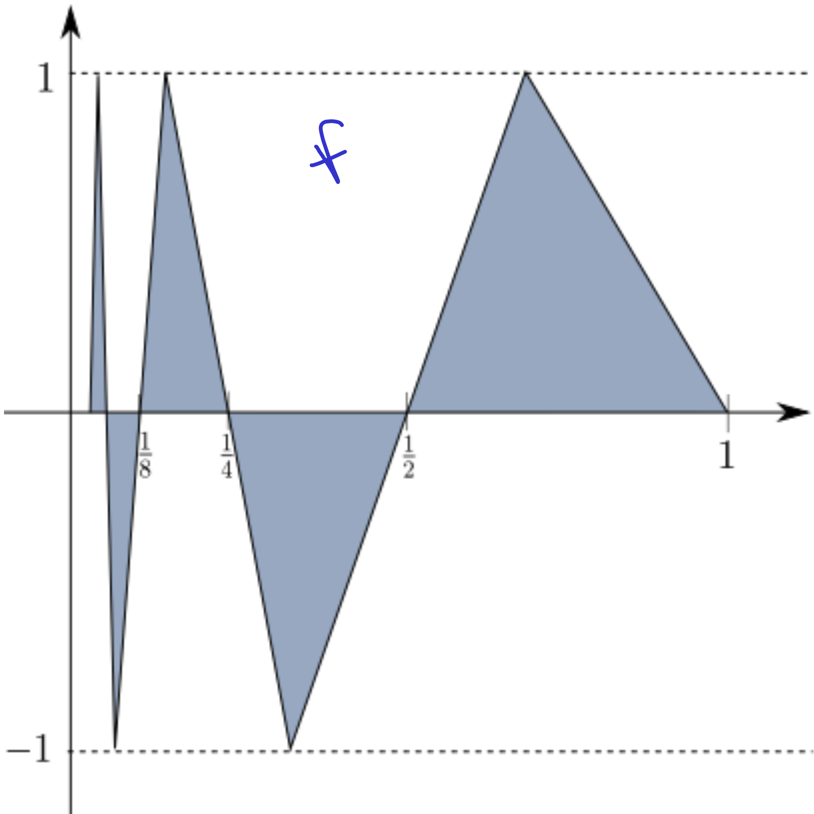
Notemos que $\forall N \in \mathbb{N}^+$, $N \geq 3$:

$$\int_3^N \frac{\log(x)}{x}\,dx \underset{\substack{y=\log(x)\\ dy=\frac{1}{x}dx}}{=} \int_{\log(3)}^{\log(N)} y\,dy = \frac{y^2}{2}\Big|_{\log(3)}^{\log(N)} =$$

$$\frac{\log^2(N)}{2} - \frac{\log^2(3)}{2} \xrightarrow[N\to\infty]{} \infty.$$

$$\therefore \sum_{k=1}^{\infty} \frac{\log(k)}{k} = \infty \quad \text{(la serie diverge).}$$

Ayer probamos que $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k\cdot\log(k)}$ diverge, así que no debe sorprendernos que $\displaystyle\sum_{k=1}^{\infty} \frac{\log(k)}{k} = \infty$



$$\int_0^1 f(x)\,dx =$$

$$\frac{1}{2}\left[\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \cdots\right]$$

$$= \boxed{\frac{1}{2}} \cdot \sum_{k=1}^{\infty} (-1)^{k-1} \cdot \frac{1}{2^k} \,.$$

Dada $n \in \mathbb{N}^+$, $S_{2n} = \displaystyle\sum_{k=1}^{2n} (-1)^{k-1} \cdot \frac{1}{2^k}$

$$= \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{8} - \frac{1}{16}\right) + \cdots + \left(\frac{1}{2^{2n-1}} - \frac{1}{2^{2n}}\right)$$

$$= \left(\frac{1}{4}\right) + \left(\frac{1}{16}\right) + \cdots + \left(\frac{1}{2^{2n}}\right).$$

$$= \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n}.$$

$$= \sum_{k=1}^{n} \frac{1}{4^k}$$

Recordemos que

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r} \xrightarrow[n\to\infty]{} \frac{1}{1-r}.$$

Ent:
$$\sum_{k=0}^{\infty} \frac{1}{4^k} = \frac{1}{1-\frac{1}{4}} = \frac{1}{3/4} = \frac{4}{3}$$

$$\therefore \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{4}{3} - 1 = \frac{1}{3}.$$

Como $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2^k}$ es convergente por el criterio de Leibniz (verificar esto), entonces

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{2^k} = \lim_{n\to\infty} S_{2n} = \frac{1}{3}$$

Ent. $\int_0^1 f(x)\,dx = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$ //